Solution of Linear Equation by Matrix method.

Let $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

then $AX = B$

$[A \ B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$ is called augmented matrix of the given system of eqns.

condition for consistency

Rank of A = Rank of $[A \ B]$

If Rank of A < Rank of $[A \ B] \Rightarrow$ inconsistent

i.e. no solution.

Echelon form of a matrix: In such form,

(i) all nonzero rows must precede the zero rows

(ii) no. of zeros before Ist non-zero element in first, second, third rows must be in increasing order.

E.g.
$$\begin{bmatrix} 2 & 3 & 8 & -5 \\ 0 & 2 & -4 & 3 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Ex. Find the solution of the following system of equations

$$2x + 6y + 11 = 0$$
$$6x + 20y - 6z + 3 = 0$$
$$6y - 18z + 1 = 0$$

Soln writing the given equations as

$$2x + 6y + 0z \cancel{+11} = \cancel{0} -11$$
$$6x + 20y - 6z \cancel{+3} = -3$$
$$0x + 6y - 18z = -1$$

$\therefore A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$

i.e. $\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$

operating $R_2 \to R_2 - 3R_1$

$\Rightarrow \begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -1 \end{bmatrix}$

operating $R_3 \to R_3 - 3R_2$

$\Rightarrow \begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -91 \end{bmatrix}$

$\Rightarrow 2x + 6y = -11$

$2y - 6z = 30$

$0x + 0y + 0z = -91 \leftarrow$ is meaningless.

$\Rightarrow$ The given system of eqns is inconsistent.

Now, we find the solution of three non-homogeneous linear eqns in three variables by matrix method.

$$AX = B$$

$$\Rightarrow A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B$$

$$\Rightarrow \boxed{X = A^{-1}B}$$

Revision for finding A^{-1} i.e. inverse of a matrix.

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Then, $A^{-1} = \frac{1}{\Delta}\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

where $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ are cofactors of $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$.

Q. Solve

$$x + y + z = 6$$
$$2x + y - 3z = -5$$
$$3x - 2y + z = 2.$$

Soln Here, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ 3 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ -5 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -1 & -4 \\ 3 & -5 & 3 \end{vmatrix} = -23$$

Now, we find cofactors.

$$A_1 = \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} = -5, \quad A_2 = -3, \quad A_3 = -4,$$

$$B_1 = -11, \quad B_2 = -2, \quad B_3 = 5,$$

$$C_1 = -7, \quad C_2 = 5, \quad C_3 = -1$$

$$\therefore A^{-1} = \frac{1}{|A|}\begin{bmatrix} -5 & -3 & -4 \\ -11 & -2 & 5 \\ -7 & 5 & -1 \end{bmatrix} = \frac{-1}{23}\begin{bmatrix} 5 & 3 & 4 \\ 11 & 2 & -5 \\ 7 & -5 & 1 \end{bmatrix}$$

$$\therefore X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 5 & 3 & 4 \\ 11 & 2 & -5 \\ 7 & -5 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 5\times 6 + 3\times(-5) + 4\times 2 \\ 11\times 6 + 2\times(-5) + (-5)\times 2 \\ 7\times 6 + (-5)\times(-5) + 1\times 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 23 \\ 46 \\ 69 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow x = 1, \; y = 2, \; z = 3$$